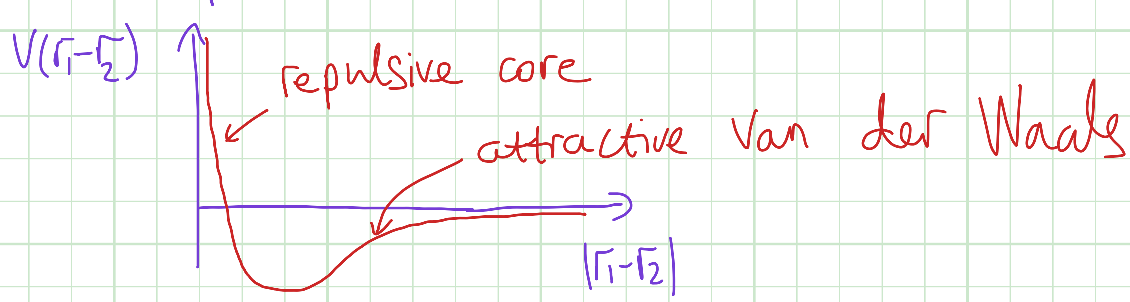
Feshbach Resonance

Last time we argued considered scattering of two particles in free space. The motion of the particles is described by the Hamiltonian

$$\mathcal{H} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(r_1 - r_2)$$

where $V(r_1 - r_2)$ is the two particle interaction. For neutral atoms at long ranges $V(r_1 - r_2)$ is dominated by the attractive van der Waals force. OTH at short ranges the interaction becomes dominated by the overlap of e^- clouds and hence repulsive.



Pseudo-potential solution:

Using spherical box potential we showed that the T-Matrix is

$$T(E) = -\frac{4\pi\hbar^2}{m} f_0 = \frac{-4\pi\hbar^2/m}{-1/a - i\sqrt{2\mu E/\hbar^2} + r_e \mu E}$$

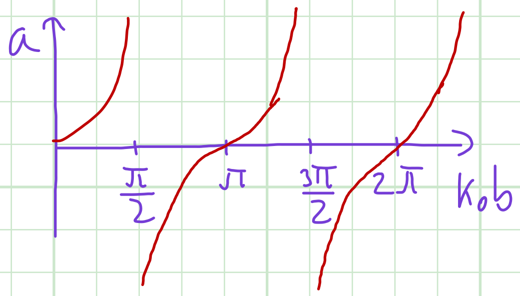
Scattering amplitude

effective range

where $a = -\tan(k_0 b)/k_0$ $\quad$ $b \Rightarrow$ radius of the sperical box potential

$$k_0 = \frac{\sqrt{2\mu(E+V_0)}}{\hbar}$$

$k_0 b = \pi\left(\frac{1}{2}+n\right) \Rightarrow$ resonance



Resonance occurs when

$$\sin(k_0 b) = \sin\left(\pi\left(\tfrac{1}{2}+n\right)\right) = \pm 1$$

Aside : Bound State energy

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} - V(r)u = Eu$$

$$u = A\sin(k_0 r) \qquad r<b$$

$$u = Ce^{-kr} \qquad r>b$$

where $k_0 = \sqrt{\frac{2\mu}{\hbar^2}(E+V_0)}$ $\quad E=-E_b<0$ for a bound state

$K = \sqrt{\frac{2\mu}{\hbar^2}(-E)}$

$$A\sin(k_0 b) = Ce^{-kb}$$

$$Ak_0\cos(k_0 b) = -Cke^{-kb}$$

$$\Rightarrow \tan(k_0 b) = -k_0/K \quad \Rightarrow \sqrt{\frac{2\mu}{\hbar^2}E_b} = -\frac{k_0}{\tan(k_0 b)}$$

to have a resonance $(E_b=0)$ we must have $\tan(k_0 b) = -\infty$

Scattering problem using Green functions $\Rightarrow$ useful for connecting the simple 2 particle scattering results to many-body physics.

Lippmann - Schwinger give a prescription for solving a QM problem of the form

$$(H_0+V)|\psi\rangle = E|\psi\rangle \qquad (*)$$

when we have a solution for the non-interacting problem

$$H_0|\phi\rangle = E|\phi\rangle$$

Consider the formal expression [this is called the Lippmann - Schwinger eq]

$$|\psi\rangle = |\phi\rangle + \frac{1}{E-H_0}V|\psi\rangle$$

$\Rightarrow$ where $\frac{1}{E-H_0}$ is symbolic notation for the operator inverse $\frac{1}{E-H_0} \equiv (E-H_0)^{-1}$

multiplying both sides of the L-S eq. by H_0-E we recover the original schrodinger eq. $(*)$. The advantage of the L-S eq. over the Schrodinger equation is that it can be used to solve for $|\psi\rangle$

iteratively in powers of V. That is set $|\psi^{(0)}\rangle = |\phi\rangle$, plug into LHS to obtain $|\psi^{(1)}\rangle = |\phi\rangle + \frac{1}{E-H_0} V|\phi\rangle$ and iterate.

We can follow a similar strategy for Green functions. A Green function is the inverse operator of the Hamiltonian

$$(\mathcal{H}-E)\,G(E) = \mathbb{1}$$

Now consider the usual setup with a part of Hamiltonian that we know how to solve H_0, and an interaction part V.

$$(\mathcal{H}_0 + V - E)\,G = \mathbb{1}$$

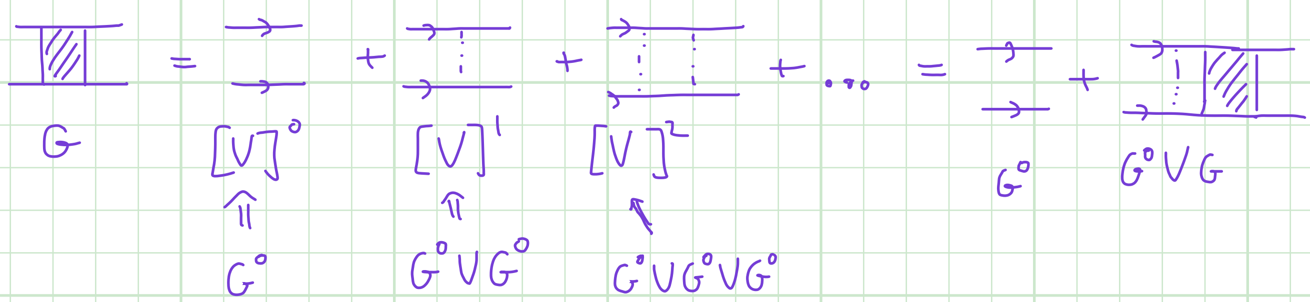
PS exp. $\quad G = \frac{1}{G_0^{-1}+V} = \frac{G_0}{\mathbb{1}+VG_0} = G_0 + G_0VG_0 + G_0VG_0VG_0 + \dots$

$$(\mathcal{H}_0 - E)\,G + VG = \mathbb{1}$$

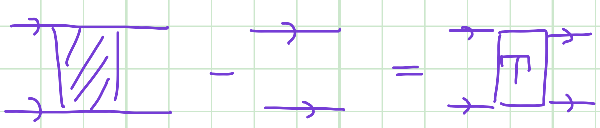
$$G = (\mathcal{H}_0-E)^{-1} - (\mathcal{H}_0-E)^{-1}VG = G_0 - G_0VG$$

[This is called the Dyson Equation]

We want to describe scattering of two particles, hence in the above formula, G represents the 2-particle Green function. The power series describes the 2 particles travelling together and scattering zero, once, twice, etc...



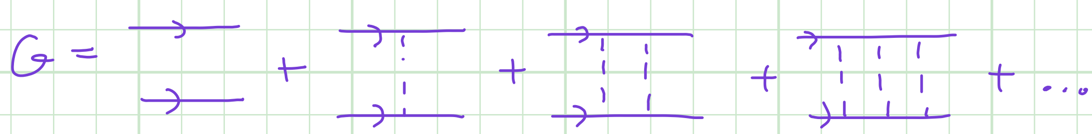
It is conventional to define the T-matrix, which is a quantity closely related to the 2-particle Green function as follows

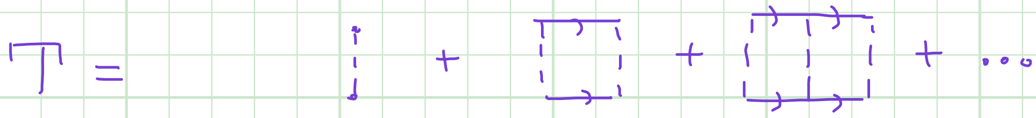


$$G \;-\; G^0 \;=\; G^0\,T\,G^0$$

That is the T-matrix does not contain the $[V]^0$, ie the free propagating part of the 2-particle Green function. It is also missing the free propagation before the 1st scattering event and after the last scattering event. In terms of diagrams comparing the

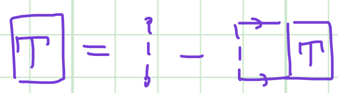
Power series expansions for the two:





Hence the L-S equation for T reads:

$$T = V - V G_0 T$$



Derivation of L-S equation for T

Starting from the def: $G_0 T G_0 = G - G_0 \Rightarrow G = G_0 + G_0 T G_0$

we plug into the L-S eq. for G: $G = G_0 - G_0 V G$

$$\Rightarrow \cancel{G_0} - G_0 T G_0 = \cancel{G_0} - G_0 V [G_0 - G_0 T G_0]$$

$$\cancel{G_0} T \cancel{G_0} = \cancel{G_0} [V - V G_0 T] \cancel{G_0}$$

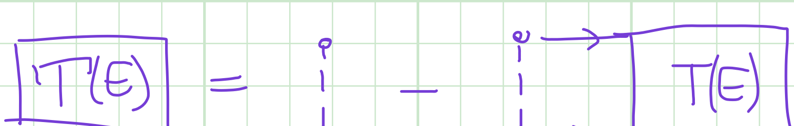
$$T = V - V G_0 T$$

Solving the L-S eq. for T

Let us focus on the case of $V(r_1 - r_2) = V_0\, \delta(r_1 - r_2)$. This form of potential is especially simple for calculations, as it becomes

$V_{kk'} = V_0$ in momentum space. However, as we shall see $V(r_1 - r_2)$ is singular, a problem that we can resolve using "regularization".

As scattering by V completely scrambles the momentum, the only quantity that T can depend on is the energy. Working in COM frame we have



V_0 $\quad V_0\, G(E)\, T(E)$

where $G(E)$ is the two particle propagator with energy E:

$$G(E) = \begin{array}{c} K,\, E/2+i\omega_n \\ \longrightarrow \\ \\ \longrightarrow \\ -K,\, E/2-i\omega_n \end{array} = \int dk \, \frac{1}{2\epsilon_K - E} = \int dk \int d\omega \; G_K\!\left(\tfrac{E}{2}+i\omega\right) G_{-K}\!\left(\tfrac{E}{2}-i\omega\right) = \int dk \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{1}{\epsilon_K - \frac{E}{2} - i\omega} \right) \left(\frac{1}{\epsilon_K - \frac{E}{2} + i\omega} \right)$$

The two particle Green function has $H-E$ in denominator where E is the pair energy and $H = 2\epsilon_K = \frac{k^2}{m}$

We can find the 2-particle G from the convolution of 1-particle G's

⇑ where we use

$$\frac{1}{2\pi} \int d\omega \left(\frac{1}{\alpha - i\omega} \right) \left(\frac{1}{\alpha + i\omega} \right) = \frac{1}{2\pi} \int d\omega \, \frac{1}{\alpha^2 + \omega^2} = \frac{1}{2\alpha}$$

Plugging this in, we obtain

$$T(E) = V_0 - V_0 \int dk \, \frac{1}{2\epsilon_K - E} \, T(E)$$

Dividing both sides by $V_0 T(E)$ we obtain

$$\frac{1}{V_0} = \frac{1}{T(E)} - \int \frac{dk}{2\epsilon_K - E} \qquad (**)$$

The integral in this expression is divergent, but it diverges in exactly the same way as the integral in the gap equation

$$\cancel{\Delta} = -V_0 \int \frac{\cancel{\Delta}}{2\sqrt{\Delta^2 + \xi_K^2}} \, dk \quad \Rightarrow \quad -\frac{1}{V_0} = \int \frac{dk}{2\sqrt{\Delta^2 + \xi_K^2}}$$

Hence we have a method for fixing the divergence in the many-body problem. We know from solving the Schrodinger equation that $T(E) = \dfrac{2\pi/\mu}{\frac{1}{a} + i\sqrt{2\mu E/\hbar^2}}$. Hence, if we set $E=0$ in Eq. $(**)$ we find

$$\frac{1}{V_0} = \frac{\mu}{2\pi a} - \int \frac{dk}{2\epsilon_K} \quad \Rightarrow \quad -\left[\frac{\mu}{2\pi a} - \int \frac{dk}{2\epsilon_K} \right] = \int \frac{dk}{2\sqrt{\Delta^2 + \xi_K^2}}$$

$$-\frac{\mu}{2\pi a} = \int dk \left[\frac{1}{2\sqrt{\Delta^2 + \xi_K^2}} - \frac{1}{2\epsilon_K} \right] \qquad (***)$$